Some generalities on Group Reps and Tensor Categories

Since many Weil cohomologies are valued in f.d. vector spaces, we expect motives to also be "finite dimensional" in some sense (not the geometric dimension), as motives are supposed to give a universal cohomology theory. [See the Master's Thesis by Stefano Nicotra for the details. Much of this lecture is from that source].

We begin with some abstract nonsense.

<u>Def</u>: A tensor category is a 5-tuple $(\mathcal{C}, \otimes, \varphi, \psi, (\mathbb{1}, e))$, where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, such that

1) associativity (pentagon axiom!) $\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \xrightarrow[\text{natural}]{\sim} (X \otimes Y) \otimes Z$
2) commutativity $\psi_{X,Y} : X \otimes Y \xrightarrow[\text{natural}]{\sim} Y \otimes X$ (hexagon axiom!)
3) identity $\mathbb{1}$ for which $X \mapsto \mathbb{1} \otimes X$ is an autoequivalence, and $e : \mathbb{1} \mapsto \mathbb{1} \otimes \mathbb{1}$.

<u>Example</u>: The category of Motives is a tensor category, indeed $M \otimes N = (X \times Y, p \times q, m+n)$ in the tensor product, and $\mathbb{1} = h_\sim(\operatorname{Spec} k) = (\operatorname{Spec} k, \mathrm{id}, 0)$ is the identity. Further, recall the duality operator $D((X_d, p, m)) = (X_d, {}^T p, d-m)$ makes $\mathrm{Mot}_\sim$ a rigid, $\mathbb{Q}$-linear, pseudo-abelian tensor category (where we extend by additivity from pure dimension d.

We will work in the category of motives, but our definitions make sense for any rigid $\mathbb{Q}$-linear pseudo-abelian $\otimes$-category. Now given $X \in \mathrm{SmProj}_k$, we know that S_n acts on $X^n = X \times \cdots \times X$ via $(x_1, \dots, x_n) \overset{\sigma}{\longmapsto} (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. Further, one can check that this is an actual group action. Fixing σ, we see that the graph $\Gamma_\sigma \subset X^n \times X^n$ gives a correspondence (over $\mathbb{Q}$) in $\mathrm{Corr}^0_\sim(X^n, X^n)$. This extends $\mathbb{Q}$-linearly to a morphism

$$\mathbb{Q}[S_n] \longrightarrow \underbrace{\mathrm{Corr}^0_\sim(X^n, X^n) = \mathrm{End}(X^n)}_{\mathbb{Q}\text{-vector space}}.$$

Hence we get a rational representation of S_n. By Maschke's Theorem (as $\operatorname{char} \mathbb{Q} = 0$), we see that this representation will split into irreducibles.

Let $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \le \cdots \le \lambda_k$ be a partition of n, then we know irreps of S_n correspond bijectively to partitions. Indeed since $\mathbb{Q}[S_n]$ is semisimple:

$$\mathbb{Q}[S_n] \cong \bigoplus_{|\lambda| = n} \mathrm{End}_{\mathbb{Q}} V_\lambda$$

where V_λ is the irrep corresponding to λ. Further, each summand comes with an idempotent operator e_λ in which $e_\lambda = \mathrm{id}$ on V_λ and 0 otherwise (see Fulton-Harris or Serre for the explicit form of this idempotent).

So then given any of the e_λ, Γ_{e_λ} is a projector, as $e_\lambda \circ e_\lambda = e_\lambda$ on X^n. One can check that if $M = (X, p, m)$, $M^{\otimes n} = (X^n, p^n, nm)$, then for any $\sigma \in S_n$ we have $\Gamma_\sigma \circ p^n = p^n \circ \Gamma_\sigma$, so this does give a morphism in Mot.

Now if $\lambda = (n)$, then the corresponding projector is the graph of $\frac{1}{n!} \sum \sigma$, which is exactly the symmetrization. Hence we set

$$\mathrm{Sym}^n(M) = (X^n, \Gamma_{(n)} \circ p^n, nm).$$

If $\lambda = (1,1,\dots,1)$, then the projector is the image of $\frac{1}{n!}\sum(\operatorname{sign}\sigma)\sigma$. Which is the alternating operator. Hence we set

$$\Lambda^n(M) = (X^n, \Gamma_{(1,\dots,1)} \circ p^n, nm).$$

Back to Weil Cohomology

Let H be a Weil cohomology theory. Then recall that for $a \in H^i(X)$, $b \in H^j(X)$, $a \cup b = (-1)^{ij}\, b \cup a$. This is the condition of super-commutativity. Then we can decompose $H^\bullet(X)$ as

$$H^\bullet(X) = \underbrace{H^+(X)}_{\oplus H^{2i}(X)} \oplus \underbrace{H^-(X)}_{\oplus H^{2i+1}(X)}$$

to get a superalgebra over F. Given two super vector spaces, V_1, V_2 the tensor product is defined as

$$V_1 \otimes V_2 = \underbrace{(V_1 \otimes V_2)_{even}}_{(V_1^+ \otimes V_2^+) \oplus (V_1^- \otimes V_2^-)} \oplus \underbrace{(V_1 \otimes V_2)_{odd}}_{(V_1^+ \otimes V_2^-) \oplus (V_1^- \otimes V_2^+)}.$$

Combining this with the normal construction of symmetric and exterior powers, we get once we use Künneth:

Lemma Let $M = (X, p, 0)$. Then:

- $H(\mathrm{Sym}^n(M)) = \bigoplus_{i+j=n} \mathrm{Sym}^i H^+(M) \otimes \Lambda^j H^-(M)$.
- $H(\Lambda^n(M)) = \bigoplus_{i+j=n} \Lambda^i H^+(M) \otimes \mathrm{Sym}^j H^-(M)$

This obviously implies the following: If $H^+(M) = 0$, then $H(\mathrm{Sym}^n M) = \Lambda^n H^-(X)$, and hence is zero if $n > \dim H^-(M)$. Conversely if $H^-(M) = 0$ then $H(\Lambda^n M) = \Lambda^n H^+(X)$ and hence is zero if $n > \dim H^+(M)$.

Dimension

Inspired by the above, we have the following definition:

Def: A motive (X, p, m) is
1) evenly finite dimensional if $\exists\, n > 0$ s.t. $\Lambda^n M = 0$, i.e. $\Gamma_{(1,\dots,1)} \circ p^n \sim 0$. We set $\dim M = \max\{n \mid \Lambda^n M \neq 0\}$.
2) oddly finite dimensional if $\exists\, n > 0$ s.t. $\mathrm{Sym}^n M = 0$, i.e. $\Gamma_{(n)} \circ p^n \sim 0$. We set $\dim M = \max\{n \mid \mathrm{Sym}^n M \neq 0\}$.
3) finite dimensional if there is a direct sum $M = M_+ \oplus M_-$ with M_+ evenly f.d. and M_- oddly f.d. Then $\dim M = \dim M_+ + \dim M_-$.

Remarks:

1) Although it is not obvious, the dimension is independent of the decomposition.

2) The index m never appears in the definition. Hence $(X, p, 0)$ is finite dimensional iff (X, p, m) is, and they have the same dimension.

3) Since $S_n \subset S_{n+1}$, $e_{(n+1)} = r \cdot e_{(n)}$ for some $r \in \mathbb{Q}[S_{n+1}]$, and similar for $e_{(1,\dots,1)}$.
Thus:

$$\begin{aligned} \Gamma_{(n+1)} \circ p^{n+1} &= \Gamma_r \circ (\Gamma_{(n)} \times id) \circ p^{n+1} \\ &= \Gamma_r \circ (p^n \times p) \circ (\Gamma_{(n)} \times id) \end{aligned}$$

so $\mathrm{Sym}^n \mathcal{M} = 0 \Rightarrow \mathrm{Sym}^{n+1} \mathcal{M} = 0$, and similar for $\Lambda^n \mathcal{M}$.

4) Sums, tensor product, and duals of f.d. motives are f.d.

Examples:

- $\mathbb{1} = (\mathrm{Spec}\, k, id, 0)$ is evenly one dimensional. Indeed:

$$\Lambda^2 \mathbb{1} = (\mathrm{Spec}\, k \times \mathrm{Spec}\, k, id - id, 0) = 0$$

Also $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times e, 0) \cong (\mathrm{Spec}\, k, id, -1)$ is also. We will see later that $\dim M \otimes N$ is $\leq \dim M \cdot \dim N$. Hence $h_0(X_d) = (X_d, e \times X, 0) \cong \mathbb{1}$ and $h_{2d}(X_d) = (X, X \times e, 0) \cong \mathbb{L}^{\otimes d}$ are also evenly 1 dimensional.

- Curves. Recall that if C is a sm. proj. curve, we have

$$ch(C) = ch^0(C) \oplus ch^1(C) \oplus ch^2(C).$$

we can take $ch(C)_+ = ch^0(C) \oplus ch^2(C)$ as those are evenly fin. dim. by the above, but now we have:

Thm: $ch^1(C)$ is oddly f.d. of dimension $2g$.

Proof: Since one can check that $H^1(ch^1(C)) = H^1(C)$, we have that

$$H(\mathrm{Sym}^{2g} ch^1(C)) = \Lambda^{2g} H^1(C) \neq 0.$$

hence $\dim(ch^1(C)) \geq 2g$. The goal is then to show $\mathrm{Sym}^{2g+1} ch^1(C) = 0$. Set:

$$\alpha_n = \Gamma_{(n)} \circ p_1^n = \left(\frac{1}{n!} \sum \Gamma_\sigma\right) \circ p_1^n \qquad (p_1 = \Delta - p_0 - p_2)$$

Then $\mathrm{Sym}^n ch^1(C) = (C^n, \alpha_n, 0)$, so we then need $\alpha_{2g+1} \sim 0$. Now since $\dim C = 1$, $S^n C = C^n / S_n$ is smooth with quotient map φ_n. Define:

$$\beta_n = \frac{1}{n!} (\varphi_n)_* \circ \alpha_n \circ \varphi_n^* \in CH^n(S^n C \times S^n C).$$

Step I: $\alpha_n = 0 \iff \beta_n = 0$

Step II: If $n > 2g-2$, it's well known that the Abel-Jacobi map $\pi : S^nC \to J(C)$ is a projective bundle with fibers of dimension $m = n-g$. Then setting $\xi_n = \mathcal{O}(1)$, one can show

$$CH(S^nC) \cong CH(J(C))[1, \xi_n, \dots, \xi_n^m].$$

Thus showing $\beta \in CH(S^nC)$ is zero reduces to showing that $\pi_*(\beta\,\xi_n^{\lambda}) = 0$ for all λ.

Step III: We have two projections $p_1, p_2 : S^{2g+1}C \times S^{2g+1}C \to S^{2g+1}C$. Then

$$\beta_{2g+1} = \sum_{i,j=0}^{g+1} (\pi \circ pr_1 \times \pi \circ pr_2)^* a_{ij} \cdot pr_1^* \xi^{g+1-i} \cdot pr_2^* \xi^{g+1-j} \qquad a_{ij} \in CH^{i+j-1}(J(C) \times J(C))$$

its then enough to show $a_{ij} = (\pi \circ pr_1 \times \pi \circ pr_2)_* (pr_1^* \xi^i \cdot pr_2^* \xi^j \, \beta_{2g+1}) = 0$.

a_{00} is trivial as $CH^{-1} = 0$ (note β_{2g+1} is dimension $2g+1$, so this fails when $n \leq 2g+1$). Applying the projection formula, its enough to show

$$pr_1^* \xi \cdot \beta_{2g+1} = 0.$$

Its a long computation from here.